CHARACTER VARIETIES AND HARMONIC MAPS TO ℝ-TREES

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ABSTRACT. We show that the Korevaar-Schoen limit of the sequence of equivariant harmonic maps corresponding to a sequence of irreducible $SL_2(\mathbb{C})$ representations of the fundamental group of a compact Riemannian manifold is an equivariant harmonic map to an ℝ-tree which is minimal and whose length function is projectively equivalent to the Morgan-Shalen limit of the sequence of representations. We then examine the implications of the existence of a harmonic map when the action on the tree fixes an end.

1. INTRODUCTION

For a finitely generated group Γ, Morgan and Shalen [MS] compactified the character variety of equivalence classes of $SL_2(\mathbb{C})$ representations of Γ with projective limits of the length functions associated to the representations. These limits turn out to be projectively equivalent to the length functions of actions of Γ by isometries on ℝ-trees. This built on earlier work of Culler and Shalen [CS] who had identified the ideal points of a complex curve of $SL_2(\mathbb{C})$ representations with actions on a simplicial tree. The tree arose from the Bass-Serre theory applied to the function field of the curve with the discrete valuation corresponding to the point in question.

For unbounded sequences of discrete and faithful representations, Bestvina [B] and Paulin [P] obtain an ℝ-tree whose length function is in the Morgan-Shalen class and which appears as the Gromov limit of convex hulls in $\mathbb{H}^3$. The limit involves rescaling the metric on the hulls by the maximum distortion at the center of the representation. Cooper [C] extends Bestvina's construction to obtain such a tree for sequences of representations that are not necessarily discrete and faithful. Moreover, he uses length functions to show that if the sequence eventually lies on a complex curve in the representation variety, then the limiting tree is in fact simplicial.

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In this paper we produce an $\mathbb{R}$-tree for any unbounded sequence of irreducible representations of the fundamental group of compact Riemannian manifolds along with an equivariant harmonic map from the universal cover to the tree. The starting point for this is to regard representations of the fundamental group as flat connections on $SL_2(\mathbb{C})$ bundles, or equivalently, as harmonic maps from the universal cover to $\mathbb{H}^3$. We first observe that the non-existence of any convergent subsequence of a given sequence of representations is equivalent to the statement that the energy of the corresponding harmonic maps along all such subsequences is unbounded. Then, after rescaling by the energy, the recent work of Korevaar and Schoen [KS2] applies: The harmonic maps pull back the metric from $\mathbb{H}^3$ to a sequence of (pseudo) metrics which, under suitable conditions, have a subsequence that converges pointwise to the pull back of a metric on some non-positive curvature (NPC) space. In our case, the conditions are met thanks to the Lipschitz property of harmonic maps. We then show that the NPC space is an $\mathbb{R}$-tree with length function in the projective class of the Morgan-Shalen limit. In this way, rescaling by energy turns out to be strong enough to give convergence, but at the same time is subtle enough to give a non-trivial limiting length function. The main result, which can also be thought of as an existence theorem for harmonic maps to certain trees, is:

The Korevaar-Schoen limit of an unbounded sequence of irreducible $SL_2(\mathbb{C})$ representations of the fundamental group of a compact Riemannian manifold is an equivariant harmonic map to an $\mathbb{R}$-tree. The image of this map is a minimal subtree for the group action, and the projective class of the associated length function is the Morgan-Shalen limit of the sequence.

For harmonic maps into hyperbolic manifolds, Corlette [C1], [C2] and Labourie [L] have shown that the existence of a harmonic map implies that the action is semi-simple. A partial analogue of this is our Theorem 5.3 in the last section. This also suggests that there is no obvious generalization for trees of Hartman's result [H] about uniqueness of harmonic maps. Further results in this direction for surface groups are discussed in [DDW].

We also note that this paper contains the work of Wolf [W] for surfaces. In the case considered there, sequences of discrete, faithful $SL_2(\mathbb{R})$-representations give equivariant harmonic maps $\mathbb{H}^2 \to \mathbb{H}^2$. The limiting tree then arises as the leaf space of the measured foliation coming from the sequence of Hopf differentials. A generalization of this point of view to $SL_2(\mathbb{C})$ also appears in [DDW].

2. Background

A. Trees and Lengths: Recall that an $\mathbb{R}$-tree is a metric space where any two points can be joined by a unique arc, and this arc is isometric to an interval in $\mathbb{R}$. For example, an increasing union of simplicial trees is an $\mathbb{R}$-tree. Given a representation ρ of a group Γ to the isometry group of an arbitrary metric space X, the length function is defined as

$$l_\rho(g) = \inf_{x \in X} d_X(x, \rho(g)x).$$

In the case when X is an $\mathbb{R}$-tree, we will use the fact that the length function is identically zero if and only Γ has a fixed point (see [CM]). Also recall that length functions determine actions on trees, except in some degenerate cases: Suppose that two different actions of Γ on two trees have the same, non-abelian length function (*non-abelian* means that $l(g)$ is not of the form $|h(g)|$ for some homomorphism $h : \Gamma \to \mathbb{R}$). Then there is an equivariant isometry between the respective (unique) minimal subtrees of the same length function [CM].

B. Character Variety: Now take Γ to be a finitely generated group. $\chi(\Gamma)$ will be its character variety, i.e. the space of characters of representations of Γ into $SL_2(\mathbb{C})$. Whereas the space of representations is merely an affine algebraic set and $\chi(\Gamma)$ is a closed algebraic set, the components of $\chi(\Gamma)$ containing classes of irreducible representations are affine varieties (closed, irreducible algebraic sets). Conjugacy classes of irreducible representations are characterized by their character (see [CS]).

Given a representation $\rho : \Gamma \to SL_2(\mathbb{C})$, for each g in Γ the image $\rho(g)$ acts on $\mathbb{H}^3$ by isometries. Consider the corresponding length function

$$l_\rho(g) = \inf_{x \in \mathbb{H}^3} d_{\mathbb{H}^3}(x, \rho(g)x).$$

We may think of l_ρ as a function on the generators $\gamma_1, ..., \gamma_r$ of Γ. For C the set of conjugacy classes of Γ, the Morgan-Shalen compactification of the character variety is obtained by adding the projective limits of all $\{l_\rho(\gamma), \gamma \in C\}$ in the projective space $\left(\prod[0, \infty) \setminus 0\right) / \mathbb{R}^+$ (see [MS]). Strictly speaking, [MS] uses the traces to define the compactification. That this is equivalent to using length functions follows from the fact that if $\text{tr}\rho(g) \geq 1$ then $|l_\rho(g) - 2\ln\text{tr}\rho(g)| \leq 2$, see [C]. Explicitly, given a sequence of representations $\rho_n : \Gamma \to SL_2(\mathbb{C})$, only one of the following can occur:

1. For some subsequence n', all traces $\rho_{n'}(\gamma_i)$ are bounded. Then $\rho_{n'}$ converges (possibly after passing to a further subsequence) in $\chi(\Gamma)$ (see [CL]).

2. For every subsequence n' there is some i such that $\mathrm{tr}\rho_{n'}(\gamma_i) \to \infty$ as $n' \to \infty$. Then there is an $\mathbb{R}$-tree and a representation $\rho : \Gamma \to Iso(T)$ such that l_ρ is not identically zero and $l_{\rho_n} \to l_\rho$ projectively (possibly after passing to a subsequence).

C. Harmonic maps: Recall from [D] and [C1] that given a compact Riemannian manifold M and an irreducible representation ρ of $\Gamma = \pi_1(M)$ into $SL_2(\mathbb{C})$, any ρ-equivariant map from the universal cover $\widetilde{M}$ to $\mathbb{H}^3$ can be homotoped via the heat flow to a ρ-equivariant harmonic map from the universal cover $\widetilde{M}$ to $\mathbb{H}^3$:

$$u : \widetilde{M} \to \mathbb{H}^3, \quad u(gx) = \rho(g)u(x), \quad D^*du = 0,$$

for $du : T\widetilde{M} \to T\mathbb{H}^3$ and D the pull-back of the Levi-Civita connection on $\mathbb{H}^3$. This harmonic map minimizes the energy on M

$$\int_M |du|^2 dM$$

amongst all equivariant maps $v : \widetilde{M} \to \mathbb{H}^3$ (see p. 643 of [KS1]) and is unique up to $\mathbb{R}$-translations in some complete, totally geodesic submanifold $Y \times \mathbb{R}$ of the target (see 4.4.B of [GP]).

Conversely, given a ρ-equivariant harmonic map u the representation ρ can be recovered as the holonomy of the flat connection $A + i\Phi$, for A the pull-back by u of the Levi-Civita connection on $T\mathbb{H}^3$ and $\Phi = -du/2$. Hence, $E(u) = 4\|\Phi\|_2^2$, where $\| \cdot \|_2$ denotes the L^2 norm. Flatness and harmonicity now become the equations:

$$(1) \qquad\qquad F_A = \frac{1}{2}[\Phi, \Phi], \quad d_A\Phi = 0, \quad d_A^*\Phi = 0.$$

We shall refer to Φ as the *Higgs field* of the representation ρ. Thinking of representations as flat connections allows us to see easily that a sequence escapes to infinity only if the energy of u blows up:

Proposition 2.1. *Let ρ_i be a sequence of representations of Γ with Higgs fields Φ_i. If the energy of the associated harmonic maps u_i is bounded then there is a representation ρ and a subsequence i' such that $\rho_{i'} \to \rho$ in $\chi(\Gamma)$.*

Proof. Recall that any harmonic map $u : \widetilde{M} \to \mathbb{H}^3$ (or any target of non-positive curvature) has the following property: For any x in $\widetilde{M}$ and $R > 0$ there is constant $C(R)$ independent of u such that whenever $d(x, y) < R$,

$$(2) \qquad\qquad |\nabla u|^2(y) \le C(x, R)E_{B(x,R)}(u).$$

This follows from the Bochner formula for $|\nabla u|^2$ when the target has negative curvature, see [S]. Therefore $E(u_i) \leq B$ implies uniform bounds on $\|\Phi_i\|_{C_0}$. By the first of equations (1) this implies uniform bounds on $\|F_{A_i}\|_{C_0}$ and hence uniform bounds on $\|F_{A_i}\|_p$ for any p. By standard application of Uhlenbeck's weak compactness theorem and elliptic regularity the result follows. $\square$

D. Korevaar-Schoen compactness: The previous section shows that in order to examine the ideal points of the character variety one has to look at equivariant harmonic maps of arbitrarily high energy. For this, we recall the construction in [KS2].

Let Ω be a set and u a map into an NPC space X. Use u to define the pull-back pseudometric on Ω, $d_u(x, y) = d_X(u(x), u(y))$, for any x and y in Ω. To achieve convergence in an NPC setting, some convexity is needed. Korevaar and Schoen achieve this by enlarging (Ω, d_u) to a space Ω_∞ obtained by adding the segments joining any two points in Ω, the segments joining any two points on these segments, and so on. Then they extend the pull-back pseudometric from Ω to Ω_∞ linearly. After identifying points of zero pseudodistance in (Ω_∞, d_u) and completing, one obtains a metric space (Z, d_u) isometric to the convex hull $C(u(\Omega))$ in the target X (and hence NPC).

It is a crucial point that certain inequalities (which carry over to pointwise limits) satisfied by pull-back pseudodistances are enough for this Z to be an NPC space, regardless of the pseudodistance being a pull-back or not (see Lemma 3.1 of [KS2]). The following summarizes the main results from [KS2] needed here when $\Omega = \widetilde{M}$ is the universal cover of a compact Riemannian manifold:

Theorem 2.2. *Let $u_k : \widetilde{M} \to X_k$ be a sequence of maps on the universal cover of some Riemannian manifold $\widetilde{M}$ such that:*

1. *Each X_k is an NPC space;*
2. *The u_k's have uniform modulus of continuity: For each x there is a monotone function $\omega(x,)$ so that $\lim_{R \to 0} \omega(x, R) = 0$, and $\max_{B(x,R)} d_{u_k}(x, y) \leq \omega(x, R)$.*

Then

1. *The pull-back pseudometrics d_{u_k} converge (possibly after passing to a subsequence) pointwise, locally uniformly, to a pseudometric d_∞;*
2. *The Korevaar-Schoen construction for $(\widetilde{M}, d_\infty)$ produces a metric space (Z, d_∞) which is NPC;*
3. *If the $u_k : \widetilde{M} \to X_k$ have uniformly bounded energies and are energy minimizers then the projection $u : (\widetilde{M}, d_\infty) \to Z$ is also energy minimizer;*

4. *If the u_k's are equivariant then u is also equivariant;*

5. *If $\lim\limits_{k\to\infty} E(u_k)$ is not zero then u is not trivial.*

Proof. 1. and 2. are contained in Proposition 3.7 of [KS2]. Given the locally uniform convergence, 3., 4. and 5. follow from Theorem 3.9 of [KS2]. □

3. Main Results

In this section, we let Γ be the fundamental group of a compact Riemannian manifold M, and let ρ_k be a sequence of irreducible representations of Γ in $SL_2(\mathbb{C})$ with no convergent subsequence in $\chi(\Gamma)$ (henceforth, an *unbounded* sequence). Let $u_k : \widetilde{M} \to \mathbb{H}^3$ be the corresponding harmonic maps. Rescale the metric $d_{\mathbb{H}^3}$ on $\mathbb{H}^3$ to

$$\hat{d}_{\mathbb{H}^3,k} = \frac{d_{\mathbb{H}^3}}{E(u_k)^{1/2}}$$

before pulling it back via u_k to $\tilde{d}_k(x,y) = \hat{d}_{\mathbb{H}^3,k}(u_k(x), u_k(y))$ on $\widetilde{M}$. Continue to denote the harmonic maps into the rescaled targets by u_k. With this understood, the first result of this paper may be stated as follows:

Theorem 3.1. *Let M be a compact Riemannian manifold and ρ_k an unbounded sequence of irreducible $SL_2(\mathbb{C})$ representations of $\Gamma = \pi_1(M)$. Then the Korevaar-Schoen limit of the rescaled harmonic maps $u_k : \widetilde{M} \to \left(\mathbb{H}^3, \hat{d}_{\mathbb{H}^3,k}\right)$ is an energy minimizing $u : \widetilde{M} \to T$, for T an $\mathbb{R}$-tree. In addition, Γ acts on T without fixed points, and u is Γ-equivariant.*

Proof. According to 1. of Theorem 2.2, to show convergence of the u_k's it is enough to show that they have uniform modulus of continuity. Recall the estimate (2). With this, the rescaled sequence u_k satisfies the uniform modulus of continuity condition for $\omega(x, R) = R$ for all x: $\tilde{d}_k(u_k(x), u_k(y)) \leq d_M(x,y)$. Therefore the pseudodistances $\tilde{d}_k$ converge pointwise and locally uniformly to a limiting pseudodistance $\tilde{d}$ on $\widetilde{M}$. Let u be the projection $u : \widetilde{M} \to Z$ for Z the NPC metric space obtained from $(\widetilde{M}, \tilde{d})$ by identifying points of zero distance and completing. Then u is energy minimizing by 3. of Theorem 2.2 and the harmonicity of the u_k's.

To show that the limiting NPC space Z is in fact an $\mathbb{R}$-tree, we need to show that: a) Any two points in Z can be joined by a unique arc; b) Every arc in Z is isometric to a segment in $\mathbb{R}$. Part b) follows immediately from the defining property of NPC spaces (any two points can be joined by an arc isometric to a segment in $\mathbb{R}$). For a) we must show that there are no more arcs. If d_k is

the pull-back of the standard metric on $\mathbb{H}^3$ then for each k the metric space $Z_k = \left(\widetilde{M}_\infty, d_k \right) / \sim$ is, by construction, isometric to the convex hull of the image $u_k(\widetilde{M})$ in $\mathbb{H}^3$. Therefore, for any two points $x, y \in \widetilde{M}_\infty$ the geodesic segment isometric to $[u_k(x), u_k(y)]$ lies in Z_k. Suppose $z \neq x, y$ is a third point on some other arc joining x and y, and consider the geodesic triangle with vertices $u(x)$, $u(y)$, and $u(z)$ in $\mathbb{H}^3$. It is a standard fact that there is a constant C characteristic of $\mathbb{H}^3$ such that in the standard metric $d_{\mathbb{H}^3}$ any point in the interior of this triangle has distance less than C from the edges (see the proof of Theorem 3.3 of [B]). Then in the rescaled pull-back metric $\tilde{d}_k$, any point in the interior has distance less than $C/E(u_k)^{1/2}$; hence, as $E(u_k) \to \infty$ this distance becomes arbitrarily small. Therefore, all triangles become infinitely thin at the limit. This suffices to show that there can be only one arc joining any two points, see Proposition 6.3.C of [Gr] and page 31 of [GH].

In addition, because of the rescaling $E(u_k) = 1$ for all k and using 5 of Theorem 2.2, the limit u is also non-trivial. Therefore, $T = Z$ is non-trivial.

The action of Γ on $\widetilde{M}$ extends to an action on the whole of $\widetilde{M}_\infty$ (this follows from a straightforward calculation using the fact that on each segment $[x, y]$ of length d there is only one point λd away from x and $(1 - \lambda)d$ away from y). The equivariance of each u_k and u implies that there are actions σ_k on $(\widetilde{M}_\infty, \tilde{d}_k)$ and σ on $(\widetilde{M}_\infty, \tilde{d})$ by isometries. These clearly descend to the completed quotients Z_k and T. Now suppose that Γ acted on T with some fixed point t_0. Then the constant map $w(x) = t_0$ is equivariant with respect to σ and of zero energy. Then u, the energy minimizer, also has zero energy; a contradiction. Hence, there are no points on T fixed by all elements of Γ, which is equivalent to the length function l_s being non-zero. $\square$

The pointwise convergence of the pseudometrics as $k \to \infty$ implies that the length function

$$l_\sigma(g) = \inf_{x \in \Omega_\infty} d_{\mathbb{H}^3}(u(x), u(gx))$$

is the pointwise limit of the length functions

$$l_{\sigma_k}(g) = \inf_{x \in \Omega_\infty} d_{\mathbb{H}^3}(u_k(x), u_k(gx)).$$

On the other hand the u_k's came from representations ρ_k, so let l_{ρ_k} be the length function obtained by the action of the $\rho(g)$'s on $\mathbb{H}^3$

$$l_{\rho_k}(g) = \inf_{v \in \mathbb{H}^3} d_{\mathbb{H}^3}(v, \rho_k(g)v),$$

Recall that the Morgan-Shalen limit of the ρ_k's in $\chi(\Gamma)$ is the projective limit of the length functions l_{ρ_k}.

Theorem 3.2. *The length function l_σ of the action of Γ on the Korevaar-Schoen tree T is in the projective class of the Morgan-Shalen limit of the sequence ρ_k.*

Corollary 3.3. *For a length function l appearing as the Morgan-Shalen limit of irreducible elements of $\chi(\Gamma)$, there is an $\mathbb{R}$-tree T on which Γ acts by isometries with length function l, and an equivariant harmonic map $u : \widetilde{M} \to T$.*

Proof of Theorem 3.2. Since $l_\sigma = \lim_{k\to\infty} l_{\sigma_k}$, we only need to show that l_{ρ_k} and l_{σ_k} converge projectively to the same (non-trivial) limit. But

$$l_{\sigma_k}(g) = \inf_{x \in \Omega_\infty} d_{\mathbb{H}^3}(u_k(x), u_k(gx))$$

where the inf on the right hand side is over the lengths of the geodesics in $\mathbb{H}^3$ joining $u_k(x)$ to $u_k(gx)$. Now according to Lemma 2.5 of [C], and as a result of the property of thin triangles in $\mathbb{H}^3$, this geodesic contains a subgeodesic with end points A and B such that:

$$|\,||[A,B]|| - l_{\rho_k}(g)| \leq \Delta \ , \qquad d_{\mathbb{H}^3}(B, \rho_k(g)A) \leq \Delta \ .$$

This implies $d_{\mathbb{H}^3}(A, \rho_k(g)A) \leq l_{\rho_k}(g) + 2\Delta$. Now by construction there is $x \in \Omega_\infty$ such that $u_k(x) = A$, and therefore $l_{\sigma_k}(g) \leq l_{\rho_k}(g) + 2\Delta$. It also follows from the definitions that $l_{\rho_k}(g) \leq l_{\sigma_k}(g)$. Dividing each side by $E(u_k) \to \infty$, we get the same limit. Since the action of Γ on T has no fixed points, this limit is non-trivial. This completes the proof of Theorem 3.2. $\qquad\square$

4. MINIMALITY OF THE TREE

The purpose of this section is to show that the image of the equivariant harmonic map of the previous section is a minimal tree, i.e. it does not contain any proper subtree invariant under the action of Γ. The main idea is that there is not enough energy for such a subtree.

To begin, recall that given a closed subtree T_1 of an $\mathbb{R}$-tree T and a point p not in T_1, there is a unique shortest arc from p to T_1, obtained as the closure of $\gamma \setminus T_1$ for any arc γ from p to T_1 (see 1.1 of [CM]). Call the endpoint of this unique arc $\pi(p)$.

Lemma 4.1. *The map π is distance decreasing.*

Proof. Given two points p and q not on T_1, the unique arc in T joining them either contains $\pi(p)$ or not. If it does, then it also contains $\pi(q)$, since T_1 is a subtree. Therefore $d(\pi(p), \pi(q)) < d(p, q)$. If it doesn't, then $d(\pi(p), \pi(q)) = 0$. □

Lemma 4.2. *Suppose Γ acts on T. If T_1 is Γ-invariant, then π is Γ-equivariant.*

Proof. To prove equivariance note that by invariance of the subtree, if p is not in T_1 then gp is not, either. Then if $\pi(gp) \neq g\pi(p)$, by definition $d(\pi(gp), gp) < d(g\pi(p), gp)$, and since g is an isometry $d(g^{-1}\pi(gp), p) < d(\pi(p), p)$, which contradicts the definition of $\pi(p)$. □

Now suppose that $u : \widetilde{M} \to T$ is an equivariant harmonic map.

Lemma 4.3. *If T_1 is a proper subtree of T invariant under the action of Γ, and $u(\widetilde{M}) \cap T_1 \neq \emptyset$, then $u(\widetilde{M}) \subset T_1$.*

Proof. Suppose that there is a point $u(x)$ not in T_1. The image of a closed ball around x in $\widetilde{M}$, large enough to enclose a fundamental domain, consists of finitely many arcs starting from $u(x)$ in T. Therefore, there is ε such that for any w_1 and w_2 in the image of u and within distance ε of $u(x)$ we have

$$d(\pi(w_1), \pi(w_2)) \leq \frac{1}{2} d(w_1, w_2).$$

Then by the definition of the energy density

$$|du|^2(x)d\mu = \lim_{\varepsilon \to 0} \frac{1}{\omega_n} \int_{S(x,\varepsilon)} \frac{d^2(u(x), u(y))}{\varepsilon^2} \frac{d\sigma(y)}{\varepsilon^{n-1}}$$

(see [KS2], pp. 227-228) we have $|d(\pi \circ u)|d\mu \leq \frac{1}{2}|du|d\mu$ on $u^{-1}\big(B_{u(x)}(\varepsilon)\big)$. Since u is an energy minimizer, this implies that u is constant. Therefore, the image of u cannot intersect T_1; a contradiction. □

As a consequence, we have the following:

Theorem 4.4. *Let M be a compact Riemannian manifold. For $u : \widetilde{M} \to T$ a Γ-equivariant harmonic map, the image of u is a minimal subtree of T. In particular, the image of u in the Korevaar-Schoen tree in Theorem 3.1 is minimal.*

Corollary 4.5. *Suppose that the sequence of representations lies on a complex curve in the character variety. Then its Korevaar-Schoen limit is a simplicial tree.*

Proof. If the sequence or representations lies on a curve then the limit tree T contains an invariant simplicial subtree T_0, see [CS], [C]. The corollary follows by the minimality of the image of u and the uniqueness of minimal subtrees (Proposition 3.1 of [CM]). $\square$

5. ACTIONS WITH FIXED ENDS

Recall that a *ray* in a tree T is an arc isometric to $[0, +\infty)$ in $\mathbb{R}$. Two rays are said to be equivalent if their intersection is still a ray, and an equivalence class of rays is an *end*. An end is fixed under Γ if $gR \cap R$ is a ray for any g in Γ and ray R in the end. An action of Γ on an $\mathbb{R}$-tree T is said to be *semi-simple* if either T is equivariantly isometric to an action on $\mathbb{R}$, or Γ has no fixed ends.

Suppose that the action of Γ is such that it fixes some end. Given a ray R in the fixed end, let $\pi(x)$ denote the projection of any point x to R as above, and let $R_{\pi(x)}$ be the sub-ray of R starting from $\pi(x)$. Then $R_x = [x, \pi(x)] \cup R_{\pi(x)}$ is the unique ray starting from any given point x and belonging to the given end. Now define $\phi_\varepsilon : T \to T$ by taking $\phi_\varepsilon(x)$ to be the unique point of distance ε from x on R_x.

Lemma 5.1. ϕ_ε *is equivariant.*

Proof. Since the end is invariant, $gR_x = R_{gx}$, and R_{gx} starts from gx. Now the point $g\phi_\varepsilon(x)$ is on R_{gx} and is a distance ε from gx. Then by definition $\phi_\varepsilon(gx) = g\phi_\varepsilon(x)$. $\square$

Lemma 5.2. ϕ_ε *is distance decreasing.*

Proof. For x and y with R_x subray of R_y, ϕ_ε is shift by distance ε along the ray and $d(\phi_\varepsilon(x), \phi_\varepsilon(y)) = d(x, y)$. If R_x and R_y intersect along a proper subray of both, let p be the initial point of this subray. Then the arc from x to y contains p and $d(x, y) = d(x, p) + d(p, y)$. There are three cases to consider:

1. $d(x, p) \geq \varepsilon$, $d(y, p) \geq \varepsilon$. Then $d(\phi_\varepsilon(x), \phi_\varepsilon(y)) = d(x, p) - \varepsilon + d(y, p) - \varepsilon \leq d(x, y)$.
2. $d(x, p) \leq \varepsilon$, $d(y, p) \leq \varepsilon$. Then $d(\phi_\varepsilon(x), \phi_\varepsilon(y)) = |d(x, p) - \varepsilon - (d(y, p) - \varepsilon)| \leq d(x, y)$.
3. $d(x, p) \leq \varepsilon \leq d(y, p)$ (or symmetrically, $d(y, p) \leq \varepsilon \leq d(x, p)$). Then $d(\phi_\varepsilon(x), \phi_\varepsilon(y)) = (\varepsilon - d(x, p)) + (d(y, p) - \varepsilon) = d(y, p) - d(x, p) \leq d(x, y)$.

$\square$

Now, arguing as in the proof of Lemma 4.3 applied to $\phi_\varepsilon \circ u$, and using Lemma 5.2, we conclude the following:

Theorem 5.3. *Let M be a compact Riemannian manifold, T an $\mathbb{R}$-tree on which $\Gamma = \pi_1(M)$ acts minimally and non-trivially via isometries, and suppose that there is a Γ-equivariant energy minimizing map $u : \widetilde{M} \to T$. Then either the action of Γ on T is semi-simple, or u is contained in a continuous family of distinct Γ-equivariant energy minimizers.*

References

[B]   Bestvina M. Degenerations of the hyperbolic space. *Duke Math. J.* **56** (1988), no. 1, 143–161

[C]   Cooper D. Degenerations of Representations into $SL_2(\mathbb{C})$. *U.C.S.B. Preprint* (1994)

[CL]   Cooper D.; Long D. D. Roots of unity and the character variety of a knot complement. *J. Austral. Math. Soc.* (A) **55** (1993), 90–99

[C1]   Corlette, K. Flat G-bundles with canonical metrics. *J. Differential Geom.* **28** (1988), no. 3, 361–382

[C2]   Corlette, K. Archimedean superrigidity and hyperbolic geometry. *Ann. of Math.* (2) **135** (1992), no. 1, 165–182

[CM]   Culler, M.; Morgan J. Group actions on $\mathbb{R}$-trees. *Proc. London Math. Soc.* (3) **55** (1987) 571–604

[CS]   Culler, M.; Shalen, P. B. Varieties of group representations and splittings of 3-manifolds. *Ann. of Math.* (2) **117** (1983), no. 1, 109–146.

[DDW]   Daskalopoulos, G.; Dostoglou, S.; Wentworth, R. On the Morgan-Shalen compactification of the $SL(2,\mathbb{C})$ character varieties of surface groups. Preprint, 1998

[D]   Donaldson, S. K. Twisted harmonic maps and the self-duality equations. *Proc. London Math. Soc.* (3) **55** (1987), no. 1, 127–131

[GH]   Ghys, É.; de la Harpe, P. Espaces métriques hyperboliques. *Sur les groupes hyperboliques d'après Mikhael Gromov (Bern, 1988)*, 27–45, Progr. Math., **83**, Birkhäuser Boston, Boston, MA, 1990

[Gr]   Gromov, M. Hyperbolic groups. *Essays in group theory*, 75–263, Math. Sci. Res. Inst. Publ., **8**, Springer, New York-Berlin, 1987

[GP]   Gromov, M.; Pansu, P. Rigidity of lattices: an introduction. *Geometric topology: recent developments (Montecatini Terme, 1990), 39–137*, Lecture Notes in Math.,**1504**, Springer, Berlin, 1991.

[H]   Hartman, P. On homotopic harmonic maps. *Canad. J. Math.* **19** 1967 673–687

[KS1]   Korevaar, N. J.; Schoen, R. M. Sobolev Spaces and Harmonic Maps for Metric Space Targets. *Comm. Anal. Geom.* **1** (1993), no. 4, 561–659

[KS2]   Korevaar, N. J.; Schoen, R. M. Global Existence Theorems for Harmonic Maps to Non-locally Compact Spaces. *Comm. Anal. Geom.* **5** (1997), no. 2, 213–266

[L]   Labourie, F. Existence d' applications harmoniques tordues à valeurs dans les variétés à courbure négative. *Proc. A.M.S.* **111** (1991), no. 3, 877–882

[MS]   Morgan, J. W.; Shalen, P. B. Valuations, trees, and degenerations of hyperbolic structures I. *Ann. of Math.* (2) **120** (1984), no. 3, 401–476

[P]   Paulin, F. Topologie de Gromov équivariante, structures hyperboliques et arbres réels. *Invent. Math.* **94** (1988), no 1, 53–80

[S]   Schoen, R. The Role of Harmonic Mappings in Rigidity and Deformation Problems. In *Complex Geometry*, 179–200, Lecture Notes in Pure and Applied Mathematics, Volume 143, Marcel Dekker, New York, 1993.

[W]   Wolf, M. Harmonic maps from surfaces to $\mathbb{R}$-trees. *Math. Z.* **218** (1995), no. 4, 577–593

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